

Cosmological scaling solutions of minimally coupled scalar fields in three dimensions

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Abstract

We examine Friedmann–Robertson–Walker models in three space-time dimensions. The matter content of the models is composed of a perfect fluid, with a γ -law equation of state, and a homogeneous scalar field minimally coupled to gravity with a self-interacting potential whose energy density red-shifts as $a^{-2\nu}$, where a denotes the scale factor. Cosmological solutions are presented for different range of values of γ and ν . The potential required to agree with the above red-shift for the scalar field energy density is also calculated.

During the last years a great number of works have been dedicated to study three-dimensional gravity, particularly after it was shown to be a soluble system [1] and to contain black hole solutions [2]-[3]. However, previously to this results, cosmological models were considered in three-dimensional gravity. In particular, some Friedmann–Robertson–Walker (FRW) models were analyzed in [4, 5, 6]. Recently, the cosmic holographic principle has

been examined using these models [7]-[8]. In this article we extend these results studying FRW models in $2 + 1$ dimensions filled with a perfect fluid, obeying a γ -law equation of state, and a homogeneous scalar field ϕ minimally coupled to gravity with a self-interacting potential $V(\phi)$. The cases where the scalar field energy density, ρ_ϕ , redshifts as $a^{-2\nu}$, with $0 \leq \nu < 1$ (a denotes the scale factor) are the only ones considered. We find new exact solutions for some ranges of values of γ and ν and the potential required to agree with the above redshift for the scalar field energy density is calculated. The range of ν is chosen such that the scalar field contributes with a negative pressure since we are interested in the counterparts of the $3 + 1$ dimensional “quintessence models” [9],[10],[11]. These type of models has been investigated since cosmological observations indicate that there must be some kind of dark energy with negative pressure in the universe. The range considered for ν fits cosmological observations properly. Although cosmological observations are irrelevant in three dimensions, from theoretical point of view it is interesting to investigate the non-trivial features of the cosmological solutions, arising from consider universes containing two fluids, which have a positive and a negative pressure, respectively. As we shall see below, these cases, discussed in terms of the energy conditions, have important differences with respect to the $3 + 1$ dimensional ones. Matter with negative pressure also has been considered in the analysis of the holographic principle in four dimensional two-fluid universes [12].

A homogeneous and isotropic universe in three dimensions is described by the line element

$$ds^2 = -d\tau^2 + a^2(\tau) \left(\frac{dr^2}{1 - \kappa r^2} + r^2 d\phi^2 \right), \quad (1)$$

where $a(\tau)$ is the scale factor and $\kappa = -1, 0, 1$ for hyperbolic, flat and circular two-dimensional spatial geometry, respectively. The Einstein field equations of this model are

$$\frac{\dot{a}^2 + \kappa}{a^2} = K(\rho_m + \rho_\phi), \quad (2)$$

$$\frac{\ddot{a}}{a} = -K(p_m + p_\phi), \quad (3)$$

$$\ddot{\phi} + 2 \frac{\dot{a}}{a} \dot{\phi} = - \frac{\partial V(\phi)}{\partial \phi}, \quad (4)$$

where K is the gravitational constant in $2 + 1$ dimensions and the dot de-

notes a derivative with respect to τ . The energy density associated to the homogeneous scalar field $\phi(\tau)$ is given by

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi) \quad (5)$$

and the pressure by

$$p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi). \quad (6)$$

ρ_m and p_m denote the density and the pressure of the fluid, respectively.

From (2) and (3) it is possible to write

$$\dot{\rho}_m + \dot{\rho}_\phi + 2\frac{\dot{a}}{a}(\rho_m + p_m + \rho_\phi + p_\phi) = 0, \quad (7)$$

which corresponds to *total* energy-momentum tensor conservation, and handling (5) and (6) it is shown that equation (4) is equivalent to the conservation of the scalar field energy-momentum tensor

$$\dot{\rho}_\phi + 2\frac{\dot{a}}{a}(\rho_\phi + p_\phi) = 0. \quad (8)$$

This occurs because the scalar field interacts with no fields other than gravity. Thus, equations (7) and (8) imply

$$\dot{\rho}_m + 2\frac{\dot{a}}{a}(\rho_m + p_m) = 0. \quad (9)$$

Therefore, each energy-momentum tensor is conserved independently and the dynamics of our model is governed by the Friedmann equation (2) and the conservation laws (8) and (9).

We consider that the pressure p_m and the density ρ_m of the fluid are related by the γ -law

$$p_m = (\gamma - 1)\rho_m, \quad \text{with} \quad 1 \leq \gamma \leq 2. \quad (10)$$

and that the field obeys the same type of equation of state

$$p_\phi = (\nu - 1)\rho_\phi, \quad 0 \leq \nu < 1. \quad (11)$$

Note that (11) does not overdetermine the system of equations since the potential $V(\phi)$ is not specified. This potential, consistent with the above

equation of state, or equivalently with the redshift (13), will be calculated below. This procedure was first used in [13], and recently has been considered in relation to quintessence scenario (see, for instance, [14] and [15]).

Using the the energy–momentum tensor conservation law and the equation of state for each matter component, we obtain

$$\rho_m = \rho_{m_0} \left(\frac{a_0}{a} \right)^{2\gamma}, \quad (12)$$

and

$$\rho_\phi = \rho_{\phi_0} \left(\frac{a_0}{a} \right)^{2\nu}, \quad (13)$$

where the constants a_0 and ρ_{m_0} , ρ_{ϕ_0} are the scale factor and the energy densities at $\tau = 0$, respectively.

To find solutions for our model we replace the equations of state (10) and (11), and the energy density expressions (12) and (13) in equation (2). After straightforward manipulations, we obtain

$$\int_{a_0}^a \frac{dx}{\sqrt{K(\rho_{m_0} a_0^{2\gamma} x^{-2\gamma+2} + \rho_{\phi_0} a_0^{2\nu} x^{-2\nu+2}) - \kappa}} = \tau. \quad (14)$$

We note from equation (14) that ρ_{m_0} , ρ_{ϕ_0} and a_0 must be satisfy

$$K\rho_{m_0}a_0^2 + K\rho_{\phi_0}a_0^2 - \kappa \geq 0, \quad (15)$$

in order to have a real value for τ .

Integrating left-hand side of expression (14) we find the behavior of the scale factor a as a function of τ , with the initial condition $a(\tau = 0) = a_0$. The solutions are summarized in the tables 1 and 2. Table 1 includes cases with a perfect fluid with $1 \leq \gamma \leq 2$ and a cosmological constant, since the case $\nu = 0$ corresponds to a model with cosmological constant $\Lambda \equiv K\rho_{\phi_0}$. Table 2 includes the cases with a perfect fluid and a scalar field with a negative effective pressure. Only the $\kappa = 0$ flat case is shown for all values of γ and ν considered.

Models with a cosmological constant

In these cases the condition (15) becomes $K\rho_{m_0}a_0^2 + \Lambda a_0^2 - \kappa \geq 0$. If $\Lambda = 0$ this relation implies a lower bound for $\rho_{m_0}a_0^2$ given by

$$\rho_{m_0}a_0^2 \geq \frac{\kappa}{K}. \quad (16)$$

Table 1: Scale factor in the case $\nu = 0$.

case	a/a_0
$\Lambda > 0$	
$\kappa = 0, 1 \leq \gamma \leq 2$	$(\cosh \omega_{(\gamma)} \tau + A_{(0)} \sinh \omega_{(\gamma)} \tau)^{\frac{1}{\gamma}}$
$\kappa \neq 0, \gamma = 1$	$\cosh \omega_{(1)} \tau + A_{(\kappa)} \sinh \omega_{(1)} \tau$
$\kappa \neq 0, \gamma = 2$	$((1 + B_{\kappa}) \cosh \omega_{(2)} \tau + A_{(\kappa)} \sinh \omega_{(2)} \tau - B_{\kappa})^{\frac{1}{2}}$
$\Lambda < 0$	
$\kappa = 0, 1 \leq \gamma \leq 2$	$(\cos \omega_{(\gamma)} \tau + A_{(0)} \sin \omega_{(\gamma)} \tau)^{\frac{1}{\gamma}}$
$\kappa \neq 0, \gamma = 1$	$\cos \omega_{(1)} \tau + A_{(\kappa)} \sin \omega_{(1)} \tau$
$\kappa \neq 0, \gamma = 2$	$((1 + B_{\kappa}) \cos \omega_{(2)} \tau + A_{(\kappa)} \sin \omega_{(2)} \tau - B_{\kappa})^{\frac{1}{2}}$
$\Lambda = 0$	
$\gamma = 1$	$1 + (K \rho_{m_0} - \kappa a_0^{-2})^{\frac{1}{2}} \tau$
$\kappa = 0, 1 \leq \gamma \leq 2$	$(1 + \gamma (K \rho_{m_0})^{\frac{1}{2}} \tau)^{\frac{1}{\gamma}}$
$\kappa = 1, 1 < \gamma < 2$	$(\cos(\gamma - 1)\eta + (K \rho_{m_0} a_0^2 - 1)^{\frac{1}{2}} \sin(\gamma - 1)\eta)^{\frac{1}{\gamma-1}}$
$\kappa = -1, 1 < \gamma < 2$	$(\cosh(\gamma - 1)\eta + (K \rho_{m_0} a_0^2 + 1)^{\frac{1}{2}} \sinh(\gamma - 1)\eta)^{\frac{1}{\gamma-1}}$
$\kappa \neq 0, \gamma = 2$	$(1 - \kappa \tau^2 + 2(K \rho_{m_0} - \kappa a_0^{-2})^{\frac{1}{2}} \tau)^{\frac{1}{2}}$
	where $\omega_{(\gamma)} = \gamma \Lambda ^{\frac{1}{2}}, A_{(\kappa)} = (\frac{\Lambda + K \rho_{m_0} - \kappa a_0^{-2}}{ \Lambda })^{\frac{1}{2}},$
	$B_{\kappa} = \frac{\kappa a_0^{-2}}{2\Lambda}$ and $\tau = \int a(\eta) d\eta.$

The solutions for $\Lambda = 0$ were analyzed in [6] for radiation and dust. In the dust case $a(\tau) \propto \tau$ independently of the spatial curvature. This occurs since the dust matter density redshifted as a^{-2} in three dimensions and thus it can be combined with the term associated to the curvature κ , as equation 2 shows. In four dimensions, this situation occurs with texture or tangled strings, which are particular cases of minimally coupled scalar fields with an effective equation of state $p = -\rho/3$. They have an energy density which redshifted as a^{-2} , and thus mimics a negative curvature term. When $\kappa = 1$ and $\rho_{m_0} a_0^2 = \frac{\kappa}{K}$ we have a special situation, which has no counterpart in four dimensions, where the universe remains static.

In the radiation case, $\gamma = 3/2$, and $a(\tau) \propto \tau^{2/3}$ if $\kappa = 0$. In four

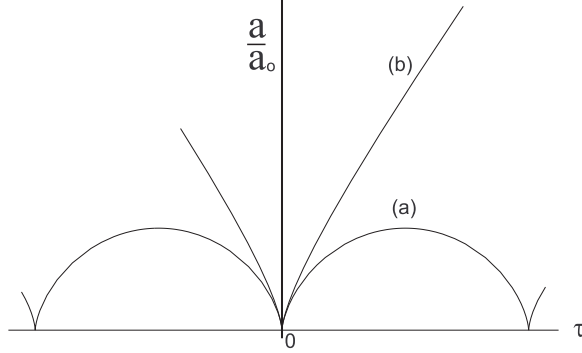


Figure 1: Scale factor $\frac{a}{a_0}$ as function of proper time τ . (a) $\kappa = 1$ closed universe and (b) $\kappa = -1$ open universe. Here $\Lambda = 0$ and $\gamma = \frac{3}{2}$.

dimensions, $a(\tau) \propto \tau^{1/2}$ for the flat case.

The figure 1(a) shows the scale factor for a closed universe a reaches a maximum value and contracts to zero in a finite proper time. The figure 1(b) shows the case $\kappa = -1$ open universe. This universe expands forever after a vanishes.

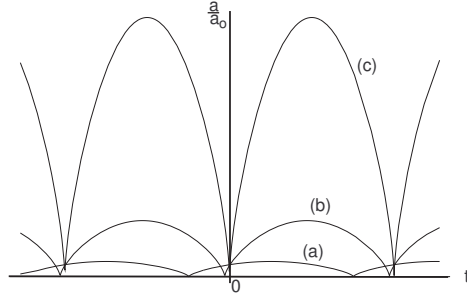


Figure 2: Scale factor $\frac{a}{a_0}$ as function of $t = \frac{3}{2}\sqrt{\Lambda}\tau$. Universes with negative cosmological constant and $\kappa = 0$. (a) $A_{(0)} = 0.1$ (b) $A_{(0)} = 10$ (c) $A_{(0)} = 100$. Here $\gamma = \frac{3}{2}$.

If $\Lambda < 0$, the solutions with $\kappa = 0$, $1 \leq \gamma \leq 2$ and $\kappa \neq 0$, $\gamma = 1$ always collapses to $a = 0$ in a finite proper time as is shown in figure 2. For the dust case, the proper time is given by

$$\tau = \frac{1}{|\Lambda|^{1/2}} \arctan \left(\left(\frac{K \rho_{m_0} a_0^2 - \kappa - |\Lambda| a_0^2}{|\Lambda| a_0^2} \right)^{-1/2} \right), \quad (17)$$

which was found in [16], in order to prove that static black hole solution in 2+1 dimensions arises naturally from gravitational collapse of pressureless dust with a negative cosmological constant. Since $\Lambda < 0$ implies an attractive cosmological force these universes begin in a singularity and end in a big crunch, i. e., they behave like the standard closed model, even with a flat or hyperbolic curvature.

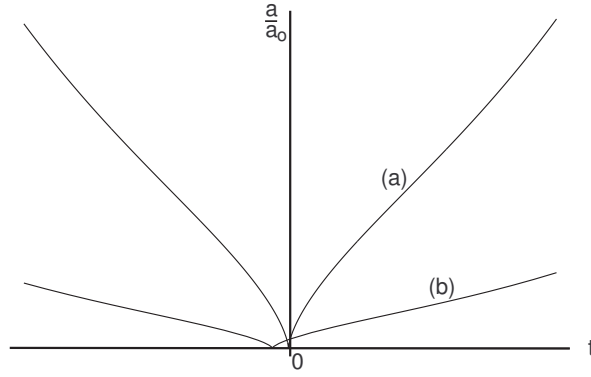


Figure 3: Scale factor $\frac{a}{a_0}$ as function of $t = \frac{3}{2}\sqrt{\Lambda}\tau$. Universes with positive cosmological constant and $\kappa = 0$. (a) $A_{(0)} = 100$ (b) $A_{(0)} = 10$. Here $\gamma = \frac{3}{2}$.

If $\Lambda > 0$, the solutions for universes filled with dust, are always inflationary-type solutions and the scalar field vanishes at

$$\tau = \frac{1}{2\sqrt{\Lambda}} \ln \frac{A_{(\kappa)} - 1}{A_{(\kappa)} + 1} \quad (18)$$

if $A_{(\kappa)} > 1$, that is, if $K\rho_{m_0} > \kappa a_0^{-2}$. In the flat case and $1 < \gamma \leq 2$, there are no restrictions and a always vanishes at a proper time given by a expression similar to (18).

Models with scalar field

From the expression $\dot{\phi}^2 = \rho_\phi + p_\phi = \nu\rho_\phi$, and the Friedmann equation (2), we obtain

$$\frac{d\phi}{da} = \frac{1}{a} \sqrt{\frac{\nu\rho_\phi}{K(\rho_\phi + \rho_m)}} \quad (19)$$

Table 2: Scale factor in the cases $\nu \neq 0$

case	a/a_0
$\kappa = 0$	
$0 < \nu < 1$	$\left[\cosh \beta(\eta - \eta_0) + \left(1 + \frac{\rho_{\phi_0}}{\rho_{m_0}}\right)^{\frac{1}{2}} \sinh \beta(\eta - \eta_0) \right]^{\frac{1}{\gamma-\nu}}$ <p>where $\beta = (\gamma - \nu)\sqrt{K\rho_{\phi_0}}a_0^\nu$ and $\tau = \int a^\nu d\eta$,</p>
$1 < \gamma < 2$	
$\kappa = -1, 0, 1$	
$\nu = 1/2$	$\frac{K\rho_{\phi_0}\tau^2}{4} + \left[K(\rho_{m_0} + \rho_{\phi_0}) - \kappa a_0^{-2} \right]^{1/2} \tau + 1$
$\gamma = 1$	

for the flat case $\kappa = 0$. After replacing equations (12) and (13) into above equation, we do the integration using

$$\int \frac{dx}{x\sqrt{c^2 + x^{-n}}} = \frac{2}{cn} \operatorname{arcsinh} c x^{\frac{n}{2}}. \quad (20)$$

Thus, we obtain an explicit expression of a in terms of ϕ . Now, considering the relation $V(\phi) = (1 - \nu/2)\rho_\phi$ and equation (13), we write down the potential

$$\begin{aligned} V(\phi) &= (1 - \nu/2)\rho_{\phi_0} \left[(\cosh(\gamma - \nu)\sqrt{\frac{K}{\nu}}(\phi - \phi_0) \right. \\ &\quad \left. + \left(1 + \frac{\rho_{m_0}}{\rho_{\phi_0}}\right)^{\frac{1}{2}} \sinh(\gamma - \nu)\sqrt{\frac{K}{\nu}}(\phi - \phi_0) \right]^{\frac{-2\nu}{\gamma-\nu}}. \end{aligned} \quad (21)$$

In order to consider the model where only a scalar field energy density exists, we set $\gamma = 1$ and $\rho_{m_0} = 0$. From (21) we read the field potential in this case

$$V(\phi) = (1 - \nu/2)\rho_{\phi_0} \exp(-2\sqrt{K\nu}(\phi - \phi_0)). \quad (22)$$

With this potential we find a special solution for which

$$a(\tau) = a_0[1 + \nu\sqrt{K\rho_{\phi_0}}\tau]^{1/\nu} \quad \text{and} \quad \phi(\tau) = \frac{1}{\sqrt{K\nu}} \ln |1 + \nu\sqrt{K\rho_{\phi_0}}\tau| + \phi_0.$$

In [5] this potential, with $\nu = 1/2$, was used and a different solution was found (but with the same asymptotic behavior). This solution does not belong to the class of solutions that satisfies $p_\phi/\rho_\phi = \text{constant}$.

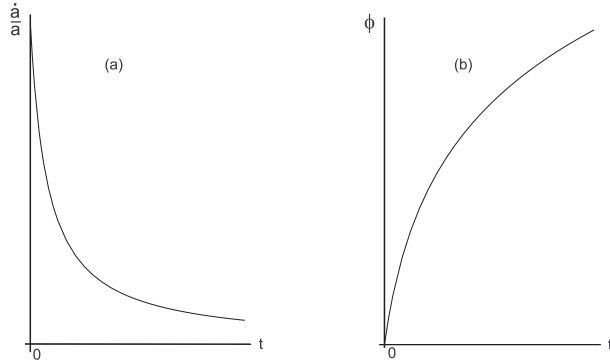


Figure 4: (a) The expansion rate of the cosmological model which only contains a scalar field only with $\nu = 1/2$. (b) The time dependence of the scalar field. Here $t = \sqrt{K\rho_{\phi_0}}\tau$ and we set $\phi_0 = 0$.

Is interesting to observe that in the cases with $\nu = 1/2$ and dust as matter component, we obtain an inflationary power-law solution.

The cosmological solutions found in this paper can be interpreted from the point of view of the specific energy conditions that guarantee geodesic convergence under gravity in 2+1 dimensions (see Barrow *et al* [5]). The *weak energy condition* (WEC) implies, for a perfect fluid, regardless of the number of spatial dimensions, the condition $\rho > 0$, that is, the matter density must be positive. The *strong energy condition* (SEC) in a $(D + 1)$ -dimensional general relativistic spacetime leads to the following restriction on ρ and p

$$(D - 2)\rho + Dp > 0. \quad (23)$$

In 2 + 1 dimensions equation (23) implies that the pressure must be positive, $p > 0$. In the (3 + 1)-dimensional case SEC implies $\rho + 3p > 0$. Therefore, in 2 + 1 dimensions WEC and SEC place independent positivity conditions on the density and pressure. Equation (3) allows us to conclude that if SEC is satisfied (in 2 + 1 dimensions) then [17]

$$\ddot{a} < 0, \quad (24)$$

which is analogous to the (3 + 1)-dimensional case. The corresponding cosmological solutions must be non-inflationary, independent of whether the universe is open, flat, or closed. SEC can be violated by a positive cosmological constant or by a scalar field with a negative mean pressure [18]. In the

models of inflation in $3+1$ dimensions, which consider an effective cosmological constant, all the solutions have $\ddot{a} > 0$. A late-time accelerated expansion of the scale factor occurs in FRW models filled with dust and some kind of energy with a negative average pressure (a positive cosmological constant or a self-interacting scalar field). In these scenarios, during the most part of the lifetime of the universe the expansion is decelerating. At very late times the acceleration becomes zero and then begins to increase.

In the $(2+1)$ -dimensional case, since the acceleration depends only on the pressure, an inflationary solution for the scale factor is obtained if the total pressure is negative. For a dust filled universe with a negative cosmological constant or a scalar field energy density the scale factor is always accelerating. Solutions with a late-time accelerated expansion are found for perfect fluids with nonzero pressure, i. e., with $1 < \gamma \leq 2$.

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